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ANALYSIS OF A THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR AN INHOMOGENEOUS TRUNCATED HOLLOW CONE†

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The three-dimensional stress–strain state of an inhomogeneous thin truncated hollow cone is studied by the method of direct asymptotic integration of the equations of the theory of elasticity [1]. Assuming that the load is sufficiently smooth, inhomogeneous solutions are constructed, which enable the load to be removed from the lateral surface of the cone. Homogeneous solutions are then constructed. Asymptotic expansions of homogeneous solutions are obtained, which enable the stress–strain state to be computed under various boundary conditions on the ends of the cone. The nature of the stress–strain state is clarified by a qualitative analysis. It is shown that, as in the homogeneous case [2], the stress–strain state consists of three types: the internal stress state, the simple boundary effect, and the boundary layer.

1. CONSIDER the axisymmetric problem of the theory of elasticity for an inhomogeneous truncated hollow cone with two conical and two spherical boundaries. We will consider the cone in a spherical system of coordinates r, θ, φ, where

$$r_1 \leqslant r \leqslant r_2, \quad \theta_1 \leqslant \theta \leqslant \theta_2, \quad 0 \leqslant \varphi \leqslant 2\pi$$

We introduce new dimensionless variables η and ρ

$$\eta = (\theta - \theta_0)/\varepsilon, \quad \rho = r/r_0$$

where $\theta_0 = (\theta_1 + \theta_2)/2$ is the angle at the vertex of the middle surface of the cone and $\varepsilon = (\theta_2 - \theta_1)/2$ is a small parameter characterizing the thickness $r_0 = (r_1 r_2)^{1/2}$ of the cone. Note that $\eta \in [-1, 1]$ and $\theta_0 \in]0, \pi/2[$.

We assume that the Lamé parameters $G = G(\eta)$ and $\lambda = \lambda(\eta)$ are arbitrary positive piecewise-continuous functions of η.

The equations of equilibrium in terms of displacements have the form

$$L\mathbf{u} \equiv (L_0 + \varepsilon\partial_1 L_1 + \varepsilon^2 \partial_1^2 L_2)\mathbf{u} = 0 \tag{1.1}$$

Here $\mathbf{u} = (u_r, \ u_\theta)'$, u_r, and u_θ are the components of the displacement vector, and L_k are matrix-valued differential operators of the form

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$$L_0 = \left\| \begin{matrix} \partial G\partial + \varepsilon G \operatorname{ctg}(\theta_0 + \varepsilon\eta)\partial - 2\kappa\varepsilon^2 & \begin{matrix} -\varepsilon\partial G - \varepsilon\kappa\partial - \\ -(G+\kappa)\varepsilon^2 \operatorname{ctg}(\theta_0 + \varepsilon\eta) \end{matrix} \\ \varepsilon\partial(\kappa+\lambda) + 2\varepsilon G\partial & \begin{matrix} \partial\kappa\partial + (2\varepsilon G + \varepsilon\partial\lambda)\times \\ \times \operatorname{ctg}(\theta_0 + \varepsilon\eta) - \kappa\varepsilon^2 \times \\ \times \csc^2(\theta_0 + \varepsilon\eta) \end{matrix} \end{matrix} \right\|$$

$$L_1 = \left\| \begin{matrix} 2\varepsilon\kappa & \lambda\partial + \partial G + \varepsilon(G+\lambda)\operatorname{ctg}(\theta_0 + \varepsilon\eta) \\ G\partial + \partial\lambda & 2\varepsilon G \end{matrix} \right\|$$

$$L_2 = \left\| \begin{matrix} \kappa & 0 \\ 0 & G \end{matrix} \right\|$$

$$\partial = \frac{\partial}{\partial\eta}, \quad \partial_1 = \rho\frac{\partial}{\partial\rho}, \quad \partial_1^2 = \rho^2\frac{\partial^2}{\partial\rho^2}, \quad \kappa = 2G + \lambda$$

We will assume that the following boundary conditions are specified on the lateral surfaces of the cone

$$\boldsymbol{\sigma}\big|_{\eta=\pm1} = M\mathbf{u}\big|_{\eta=\pm1} = \mathbf{q}^{\pm}(\rho) \tag{1.2}$$

Here

$$\boldsymbol{\sigma} = (\sigma_{r\theta}, \sigma_{\theta\theta}), \quad \mathbf{q}^{\pm}(\rho) = (f^{\pm}(\rho), h^{\pm}(\rho))$$

$$M = (M_0 + \varepsilon\partial_1 M_1)/(\varepsilon\rho)$$

$$M_0 = \left\| \begin{matrix} G\partial & -\varepsilon G \\ (\kappa+\lambda)\varepsilon & \kappa\partial + \varepsilon\lambda\operatorname{ctg}(\theta_0 + \varepsilon\eta) \end{matrix} \right\|$$

$$M_1 = \left\| \begin{matrix} 0 & G \\ \lambda & 0 \end{matrix} \right\|$$

The loads $f^{\pm}(\rho)$ and $h^{\pm}(\rho)$ given on the lateral surfaces are assumed to be sufficiently smooth functions.

2. Consider the construction of particular solutions of (1.1) that satisfy the boundary conditions (1.2), i.e. inhomogeneous solutions.

Assuming that ε is small enough and that the load on the boundaries of the cone is of order unity with respect to ε, we will use the asymptotic method of [1] to construct inhomogeneous solutions.

We will seek a solution of problem (1.1), (1.2) in the form

$$\mathbf{u} = \varepsilon^{-1}(\mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \ldots), \quad \mathbf{u}_i = (u_{ri}, u_{\theta i})^{\mathrm{T}}, \quad i = 0,1,2,\ldots \tag{2.1}$$

The substitution of (2.1) into (1.1) and (1.2) leads to a scheme, which, after integration with respect to η yields relations for the coefficients of the expansion (2.1), that enable asymptotic formulae to be obtained for the stresses. The analysis of the stress state indicates that the stresses σ_{rr} and $\sigma_{\varphi\varphi}$ are of order ε^{-1} with respect to ε, while $\sigma_{\theta r}$ and $\sigma_{\theta\theta}$ are of order unity.

3. We shall now construct homogeneous solutions. To this end, we set $\mathbf{q}^{\pm} = \mathbf{0}$ in (1.2). Finding the solutions of the homogeneous systems in the form

$$\mathbf{u}(\rho,\eta) = \rho^{z-1/2}\boldsymbol{\upsilon}(\eta), \quad \boldsymbol{\upsilon}(\eta) = (a,b)$$

after separation of variables, we arrive at the following non-self-adjoint spectral problem

$$
\begin{aligned}
&(L_0+\varepsilon(z-\tfrac{1}{2})(L_1-\varepsilon L_2)+\varepsilon^2(z-\tfrac{1}{2})^2L_2)\mathbf{v}=\mathbf{0}\\
&(M_0+\varepsilon(z-\tfrac{1}{2})M_1)\mathbf{v}|_{\eta=\pm1}=0
\end{aligned}
\tag{3.1}
$$

The homogeneous solutions corresponding to the first iteration can be obtained from the formulae for the inhomogeneous solutions by setting $\mathbf{q}^{\pm}=\mathbf{0}$. We have

$$
\begin{aligned}
&u_r^{(1)}=(4\rho)^{-1}C_0\{-2g_0G_0^{-1}\operatorname{ctg}\theta_0+\varepsilon[8\eta-8G_1G_0^{-1}+2g_1G_0^{-1}\times\\
&\times\csc^2\theta_0+4G_0^{-1}\operatorname{ctg}^2\theta_0\left(\int_{-1}^{1}2G\kappa^{-1}(G+\lambda)\int_0^{\eta}\lambda\kappa^{-1}dxd\eta-\right.\\
&\left.-\int_{-1}^{1}\lambda\kappa^{-1}\int_{-1}^{\eta}2G(G+\lambda)\kappa^{-1}dxd\eta\right)+g_0G_0^{-2}\operatorname{ctg}^2\theta_0\times\\
&\times\left(\int_{-1}^{1}2\lambda\kappa^{-1}\int_{-1}^{\eta}Gdxd\eta-g_1\right)\Bigg]+O(\varepsilon^2)\}\\
&u_\theta^{(1)}=(2\rho)^{-1}C_0\{2+\varepsilon\left[\eta g_0G_0^{-1}-\int_0^{\eta}2\lambda\kappa^{-1}dx\right]\operatorname{ctg}\theta_0+O(\varepsilon^2)\}\\
&g_k=\int_{-1}^{1}4G\kappa^{-1}(G+\lambda)\eta^kd\eta,\quad G_k=\int_{-1}^{1}G\eta^kd\eta
\end{aligned}
\tag{3.2}
$$

The eigenvalues corresponding to these solutions are $z_0=-1/2$.

We will now construct the second iteration. We will seek a solution of the form

$$
\begin{aligned}
&a^{(2)}(\eta)=\varepsilon^{1/2}(a_{20}(\eta)+\varepsilon^{1/2}a_{21}(\eta)+\ldots)\\
&b^{(2)}(\eta)=b_{20}(\eta)+\varepsilon^{1/2}b_{21}(\eta)+\ldots\\
&z=\varepsilon^{-1/2}(\alpha_0+\varepsilon\alpha_1+\ldots)
\end{aligned}
\tag{3.3}
$$

After some reduction, we obtain

$$
\begin{aligned}
&u_r^{(2)}=\varepsilon^{1/2}\rho^{-1/2}\sum_{j=1}^{4}A_jU_{rj}^{(2)},\quad u_\theta^{(2)}=\rho^{-1/2}\sum_{j=1}^{4}A_jU_{\theta j}^{(2)}\\
&U_{rj}^{(2)}=\left\{-\alpha_{0j}\eta+\alpha_{0j}^{-1}g_0^{-1}(\alpha_{0j}^2g_1-t_0\operatorname{ctg}\theta_0)+\varepsilon^{1/2}[\eta(3/2-\right.\\
&-\alpha_{0j}\alpha_{1j}\ln\rho)+\alpha_{0j}^{-2}g_0^{-1}((g_0-t_0/2)\operatorname{ctg}\theta_0-3\alpha_{0j}^2g_1/2)+\\
&\left.+(\alpha_{0j}^2g_1-t_0\operatorname{ctg}\theta_0)g_0^{-1}\alpha_{0j}^{-1}\alpha_{1j}\ln\rho]+O(\varepsilon)\right\}\exp(\varepsilon^{-1/2}\alpha_{0j}\ln\rho)\\
&U_{\theta j}^{(2)}=\left\{1+\varepsilon^{1/2}\alpha_{1j}\ln\rho+O(\varepsilon)\right\}\exp(\varepsilon^{-1/2}\alpha_{0j}\ln\rho)\\
&t_k=\int_{-1}^{1}2G\lambda\kappa^{-1}\eta^kd\eta
\end{aligned}
\tag{3.4}
$$

We obtain the following biquadratic equation for α_{0j}

$$
(g_0g_2-g_1^2)\alpha_{0j}^4+2(g_1t_0-g_0t_1)\operatorname{ctg}\theta_0\alpha_{0j}^2+(g_0^2-t_0^2)\operatorname{ctg}^2\theta_0=0
$$

It follows from (3.4) that $\sigma_{rr}^{(2)}$ and $\sigma_{\varphi\varphi}^{(2)}$ are of order unity with respect to ε, while $\sigma_{\theta r}^{(2)}$ is of order $\varepsilon^{1/2}$ and $\sigma_{\theta\theta}^{(2)}$ is of order ε.

Let us now construct the third iteration. We will seek a solution of (3.1) in the form

$$\begin{gathered} \mathbf{v}^{(3)}(\eta)=\varepsilon(\mathbf{v}_{30}(\eta)+\varepsilon\mathbf{v}_{31}(\eta)+\ldots) \\ z=\varepsilon^{-1}(\beta_0+\varepsilon\beta_1+\ldots) \\ \mathbf{v}_{3k}=(a_{3k},b_{3k})^T,\quad k=0,1,2,\ldots \end{gathered} \tag{3.5}$$

Substituting (3.5) into (3.1), we obtain a spectral problem for the initial terms of the expansion, which describes the potential solution for a plate of non-uniform thickness, which was investigated in [3].

At the next stage of asymptotic integration we obtain a boundary-value problem for $\mathbf{v}_{31}$ and β_1.

Thus, the solutions corresponding to the third iteration have the form

$$\begin{gathered} u_r^{(3)}=\rho^{-1/2}\varepsilon\sum_{k=1}^{\infty}B_kU_{rk}^{(3)},\quad u_\theta^{(3)}=\rho^{-1/2}\varepsilon\sum_{k=1}^{\infty}B_kU_{\theta k}^{(3)} \\ U_{rk}^{(3)}=\left[p_0\beta_{0k}^{-2}\psi_k''(\eta)-p_2\psi_k(\eta)+O(\varepsilon)\right]\exp(\varepsilon^{-1}\beta_{0k}\ln\rho) \\ U_{\theta k}^{(3)}=\left[-\beta_{0k}^{-3}(p_0\psi_k'')'-2\beta_{0k}^{-1}p_1\psi_k'+\beta_{0k}^{-1}(p_2\psi_k)'+O(\varepsilon)\right]\exp(\varepsilon^{-1}\beta_{0k}\ln\rho) \\ p_0=\kappa/(4G(G+\lambda)),\quad p_1=1/(2G),\quad p_2=\lambda/(4G(G+\lambda)) \end{gathered} \tag{3.6}$$

Here $\psi_k(\eta)$ is the solution of Papkovich's generalized spectral problem for the inhomogeneous case [3, 4].

4. On the basis of the above analysis we shall determine the form of the solutions obtained.

We will study the relationship between the homogeneous solutions and the principal stress vector P acting over the cross-section $r=\text{const}$. We have

$$P=2\pi r^2\int_{\theta_1}^{\theta_2}(\sigma_{rr}\cos\theta-\sigma_{r\theta}\sin\theta)\sin\theta d\theta \tag{4.1}$$

We represent the displacements in the form

$$u_r=u_r^{(1)}+\sum_{k=1}^{\infty}C_k\rho^{z_k-1/2}a_k(\eta),\quad u_\theta=u_\theta^{(1)}+\sum_{k=1}^{\infty}C_k\rho^{z_k-1/2}b_k(\eta) \tag{4.2}$$

(the displacements defined by the second and third groups of solutions are included).

For the stresses we find that

$$\begin{gathered} \sigma_{rr}=\sigma_{rr}^{(1)}+\sum_{k=1}^{\infty}C_k\rho^{z_k-3/2}Q_k(\eta),\quad \sigma_{r\theta}=\sigma_{r\theta}^{(1)}+\sum_{k=1}^{\infty}C_k\rho^{z_k-3/2}T_k(\eta) \\ Q_k(\eta)=(z_k-1/2)\kappa a_k+\lambda(2a_k+\varepsilon^{-1}b_k'+b_k\,\mathrm{ctg}(\theta_0+\varepsilon\eta)), \\ T_k(\eta)=G(\varepsilon^{-1}a_k'+z_k-3/2)b_k)\quad (k=1,2,3,\ldots) \end{gathered} \tag{4.3}$$

The terms $\sigma_{rr}^{(1)}$ and $\sigma_{r\theta}^{(1)}$ in (4.3) correspond to the eigenvalues $z_0=-1/2$.

Substituting (4.3) into (4.1), we get

$$P = 2\pi r_0^2 \varepsilon \omega_0 C_0 + 2\pi r_0^2 \varepsilon \sum_{k=1}^{\infty} C_k \rho^{z_k + 1/2} \omega_k$$

$$\omega_0 = -G_0 g_0^{-1}(t_0 + g_0)\sin 2\theta_0 + O(\varepsilon)$$

$$\omega_k = \int_{-1}^{1} [Q_k(\eta)\cos(\theta_0 + \varepsilon\eta) - T_k(\eta)\sin(\theta_0 + \varepsilon\eta)]\sin(\theta_0 + \varepsilon\eta)d\eta$$

We shall prove that $\omega_k = 0$ for all $k = 1, 2, \ldots$. To this end, we consider the following boundary-value problem

$$\begin{aligned} \sigma_{rr}\big|_{\rho=\rho_1} &= \rho_1^{z_k - 3/2} Q_k(\eta), \quad \sigma_{r\theta}\big|_{\rho=\rho_1} = \rho_1^{z_k - 3/2} T_k(\eta) \\ \sigma_{rr}\big|_{\rho=\rho_2} &= \rho_2^{z_k - 3/2} Q_k(\eta), \quad \sigma_{r\theta}\big|_{\rho=\rho_2} = \rho_2^{z_k - 3/2} T_k(\eta) \end{aligned} \tag{4.4}$$

A necessary condition for the first problem of the theory of elasticity to be solvable is that the principal vector and principal torque of all external forces should both vanish [5].

In the case under consideration the projection of the principal vector (4.4) of external forces on to the axis of symmetry $\theta = 0$ yields

$$P_k = (\rho_2^{z_k + 1/2} - \rho_1^{z_k + 1/2})\omega_k = 0$$

This latter equality is possible only if $\omega_k = 0$. Finally, we get

$$P = 2\pi r_0^2 \varepsilon C_0 \omega_0 \tag{4.5}$$

for the principal vector.

The stress state corresponding to the second and third groups of solutions is self-balanced on each cross-section $r = \text{const}$.

The solution (3.2) corresponding to the first asymptotic process defines the inner stress–strain state of the shell. The initial terms of its expansion in ε define the torque-free stress state. The stress state corresponding to (3.4) represents the boundary effects in the applied theory of shells. The initial terms of the expansion of the solution (3.4) in ε together with the leading terms of (3.2) and (2.1) can be regarded as the solutions according to the Kirchhoff–Love theory. The third asymptotic process is defined by the solutions (3.6), which have a boundary-layer form. The leading terms of (3.6) are fully equivalent to the St Venant effect for an inhomogeneous plate [3, 4].

5. Consider the problem of removing the stresses from the end surfaces of the shell. We assume that the stresses

$$\sigma_{rr}\big|_{\rho=\rho_s} = f_{1s}(\eta), \quad \sigma_{r\theta}\big|_{\rho=\rho_s} = f_{2s}(\eta), \quad (s = 1, 2) \tag{5.1}$$

are given on the spherical part of the boundary. Here $f_{1s}(\eta)$ and $f_{2s}(\eta)$ are sufficiently smooth functions satisfying the conditions of equilibrium.

As has been demonstrated, the non-self-balanced part of the load (5.1) can be removed using the penetrating solution (3.2), the relationship between C_0 and the principal vector P being given by (4.5). Henceforth we shall assume that $P = 0$. By this assumption, $C_0 = 0$.

We will seek a solution of the form (4.2). As in [2, 6], we will use the Lagrange variational principle to determine the constants C_k. In the case in hand the variational principle takes the form

$$\sum_{s=1}^{2} \rho_s^2 \int_{-1}^{1} [(\sigma_{rr} - f_{1s})\delta u_r + (\sigma_{r\theta} - f_{2s})\delta u_\theta]\Big|_{\rho=\rho_s} \times \\ \times \sin(\theta_0 + \varepsilon\eta) d\eta = 0 \tag{5.2}$$

Assuming that δC_k are independent variations, from (5.2) we obtain the infinite system

$$\sum_{k=1}^{\infty} D_{jk} C_k = h_j \quad (j = 1,2,3,\ldots) \tag{5.3}$$

of linear algebraic equations. Here

$$D_{jk} = \left(\rho_1^{z_k + z_j} + \rho_2^{z_k + z_j}\right) \int_{-1}^{1} [Q_k(\eta) a_j(\eta) + T_k(\eta) b_j(\eta)] \sin(\theta_0 + \varepsilon\eta) d\eta$$

$$h_j = \sum_{s=1}^{2} \rho_s^{z_j + 3/2} \int_{-1}^{1} \left[f_{1s}(\eta) a_j(\eta) + f_{2s}(\eta) b_j(\eta)\right] \sin(\theta_0 + \varepsilon\eta) d\eta$$

The solvability and convergence of the reduction method for the system (5.3) was proved in [7].

In view of the fact that $\sigma_{rr}^{(2)} = O(1)$ and $\sigma_{r\theta}^{(2)} = O(\varepsilon^{1/2})$, we can refine the assumptions concerning the external load.

We shall assume that $f_{1s}(\eta)$ are of order one. We decompose the shear stresses given on the spherical parts of the boundary as follows:

$$f_{2s} = f_{2s}^{(1)} + f_{2s}^{(2)} \tag{5.4}$$

$$f_{2s}^{(1)} = \frac{1}{2} \int_{-1}^{1} f_{2s} d\eta, \quad f_{2s}^{(2)} = f_{2s} - f_{2s}^{(1)}$$

It can be shown that $f_{2s}^{(1)}$ are of order $\varepsilon^{1/2}$. Then $f_{2s}^{(2)}$ can be of order unity. Thus, we get

$$f_{1s} = O(1), \quad f_{2s}^{(1)} = O(\varepsilon^{1/2}), \quad f_{2s}^{(2)} = O(1) \tag{5.5}$$

We shall seek the unknown constants A_j and B_k in the form

$$A_j = A_{j0} + \varepsilon^{1/2} A_{j1} + \varepsilon A_{j2} + \ldots \tag{5.6}$$

$$B_k = B_{k0} + \varepsilon B_{k1} + \varepsilon^2 B_{k2} + \ldots \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.3) and taking (5.5) into account, we obtain the following systems of infinite linear algebraic equations

$$\sum_{j=1}^{4} c_{ij} A_{j0} = \tau_i \quad (i = 1,2,3,4) \tag{5.8}$$

$$\sum_{k=1}^{\infty} m_{nk} B_{k0} = d_n \quad (n = 1,2,3,\ldots) \tag{5.9}$$

Here

$$c_{ij} = (\alpha_{0j} - \alpha_{0i}) g_0^{-1} [g_1(\alpha_{0j}^2 g_1 - t_0 \operatorname{ctg}\theta_0) - g_0(\alpha_{0j}^2 g_2 - t_1 \operatorname{ctg}\theta_0)] \times \\ \times \sum_{s=1}^{2} \exp(\varepsilon^{-1/2}(\alpha_{0j} + \alpha_{0i}) \ln \rho_s)$$

$$\tau_i = \sum_{s=1}^{2} \rho_s^{3/2} \int_{-1}^{1} [f_{1s}(\alpha_{0i}^{-1} g_0^{-1}(\alpha_{0i}^2 g_i - t_0 \operatorname{ctg}\theta_0) - \alpha_{0i}\eta) +$$

$$+ f_{2s}^{(1)}] d\eta \exp(\varepsilon^{-1/2} \alpha_{0i} \ln \rho_s)$$

The matrices of system (5.9) are known from the theory of inhomogeneous plates [3]. A numerical analysis of various problems has already been carried out a number of times using (5.9).

The problem of determining A_{je}, B_{kl} ($l = 1, 2, \ldots$) invariably reduces to inverting the same matrices as those for (5.8) and (5.9).

The homogeneous and inhomogeneous solutions found not only reveal the qualitative characteristics of the three-dimensional solution in the theory of inhomogeneous shells, but can also serve as an efficient formalism for solving particular boundary-value problems, as well as the basis for evaluating simplified theories.

Note that when $G = \text{const}$ and $\lambda = \text{const}$, all the solutions obtained above are exactly the same as those for a homogeneous cone [2]. In particular, as was shown in [2], the solution corresponding to the eigenvalues $z_0 = -1/2$ is identical with the Mitchell–Neuber solution for a cone [5].

Remarks. 1. When $\theta_0 \to 0$, the solutions defined by (3.4) and (3.6) become solutions for an inhomogeneous cylinder [8].

2. The case when $\theta_0 = \pi/2$ is singular and corresponds to an inhomogeneous plate of variable thickness (this case is not considered in the present paper).

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